

Transferring entangled states through spin chains by boundary-state multiplets

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Quantum spin chains may be used to transfer quantum states between elements of a quantum information processing device. A scheme discovered recently [1] was shown to have favorable transfer properties for single-qubit states even in the presence of built-in static disorder caused by manufacturing errors. We extend that scheme in a way suggested already in [1] and study the transfer of the four Bell states which form a maximally entangled basis in the two-qubit Hilbert space. We show that perfect transfer of all four Bell states separately and of arbitrary linear combinations may be achieved for chains with hundreds of spins. For simplicity we restrict ourselves to systems without disorder.

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Quantum information processing [2] relies on a number of key elements of technology, among them quantum bits and quantum gates. Since any quantum computer will contain a large number of different quantum gates and registers, information must be transferred between these elements of the computer. One possibility for that information transfer is offered by quantum spin chains, linear arrays of suitably coupled qubits. Research on quantum information transfer by spin chains started roughly a decade ago [3] and quickly developed into an active field with many contributors (see, for example, the reviews in [4]). However, most of the research up to now has focused on the transfer of single-qubit states, although the handling of entangled multi-qubit states is of primary importance in all known algorithms of quantum information processing. In this Brief Report we show how a natural extension of a single-qubit state transfer protocol [1] can be used to achieve high-fidelity transmission of arbitrary pure two-qubit states along spin chains with up to hundreds of sites.

Spin chains for quantum information transfer mostly fall into one of two classes distinguished by the degree of “engineering”, or fine-tuning, of the nearest-neighbor couplings along the chain. Perfect state transfer (PST) may be achieved if all transition frequencies generated by the spin chain Hamiltonian are commensurate and hence the time evolution of arbitrary initial states becomes periodic. In combination with spatial symmetry this enables perfect “mirroring” of initial states located at one end of the chain [5–9]. To achieve this goal, all nearest-neighbor couplings must be tuned to specific values, hence this class of chains may be called “fully engineered”. A much simpler route to good (but not perfect) state transfer is opened by modifying only the boundary couplings affecting the very first and last spins of the chain, respectively, leaving all other couplings at one and the same value [10–12]. In the limit of weak boundary couplings the system then possesses nearly degenerate (symmetric and antisymmetric) eigenstates concentrated on the boundary spins and the dynamics of these states

may be exploited for the transfer of quantum information. That class of systems may be called boundary-dominated or optimizable, since different choices of the boundary couplings may be used to adjust fidelity and speed of the state transfer. The approach suggested in [1] is an interesting hybrid between the fully-engineered and boundary-dominated schemes. As will be explained in more detail below, the temporal structure (commensurate energy spectrum leading to perfect periodicity) and the spatial structure (boundary-localized states insensitive to perturbations originating in the interior of the chain) can be optimized at the same time.

The system under consideration is a nearest-neighbor coupled spin-1/2 XX chain with spatially symmetric couplings:

$$H = \frac{1}{2} \sum_{i=1}^{N-1} J_i (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y); \quad J_i = J_{N-i} > 0, \quad (1)$$

where $\sigma_i^{x,y}$ are Pauli matrices. The total z spin component is conserved, $[H, \sum_i \sigma_i^z] = 0$, hence subspaces of fixed total z spin component can be treated separately. By the Jordan-Wigner transformation [13], the spin chain (1) can be mapped to non-interacting lattice fermions with nearest-neighbor hopping, where an up spin maps to a (spinless) fermion, while a down spin maps to an empty site. Consequently, the Hamiltonian is diagonalized once the single-particle eigenstates $|\nu\rangle$ and the corresponding eigenvalues ε_ν are known. The single-particle Hamiltonian is a symmetric tridiagonal matrix with zeros on the diagonal and the couplings J_i next to the diagonal. The special form of that matrix determines several properties of the spectrum. The ε_ν come in pairs $\pm|\varepsilon_\nu|$, the corresponding eigenvectors being related by a sign reversal of every other component. Successive eigenvectors (as ordered by energy) are alternatingly even and odd under spatial reflection. This property makes perfect state transfer possible if the ε_ν are commensurate (see, for example, [14] for details). A prominent example [15] is given by $J_i = \sqrt{i(N-i)}$, leading to an equidistant ladder of ε_ν values. However, it is even possible to

prescribe a set of eigenvalues ε_ν and find the corresponding couplings J_i by solving a structured inverse eigenvalue problem [16, 17]. There are several algorithms for solving inverse eigenvalue problems; here we follow [1] in using an algorithm by de Boor and Golub [18]. A PST chain with particularly benign properties [1] is defined by the “inverted quadratic spectrum”

$$\varepsilon_\nu = \nu(N-1 - |\nu|); \nu = -\frac{N-1}{2}, \dots, \frac{N-1}{2}. \quad (2)$$

Note that the differences between successive ε_ν are largest close to the center of the spectrum, i.e. to $\nu = 0$, for odd N , the case on which we concentrate from now on. We note in passing that the spectrum (2) superficially resembles the cosine-shaped spectrum of the homogeneous ($J_i \equiv J$) XX chain; the couplings J_i corresponding to (2), however, are roughly constant only in the central region of the chain and oscillate significantly towards the boundaries [1].

In contrast to the fully-engineered approach exemplified by the spectrum (2), the boundary-dominated approach to quantum state transfer employs a simple pattern of couplings,

$$J_1 = J_{N-1} = \alpha J; J_i = J \text{ for } i \neq 1, N-1, \quad (3)$$

where $\alpha < 1$ is an adjustable parameter. For small α a perturbation calculation shows that there are three closely spaced energy eigenvalues close to the center of the spectrum (N is odd); the corresponding eigenvectors are dominantly localized on the boundary sites and thus can be used to transfer information back and forth between the ends of the chain. Clearly, as α gets smaller, the influence of the interior spins decreases and the fidelity of the state transfer increases, but so does the transfer time which is inversely proportional to the energy splitting between the states of the dominant triplet.

The combination of PST and boundary-dominated state transfer [1] rests on the following key observation: A PST chain with spectrum (2) may be equipped with a triplet of closely spaced energies by simply “contracting” it towards the center:

$$\varepsilon'_\nu = \varepsilon_\nu - (N-3) \operatorname{sgn} \varepsilon_\nu \quad (4)$$

($\operatorname{sgn} 0$ is to be interpreted as zero). The new spectrum ε'_ν is still commensurate, ensuring PST, but the triplet of states with energies close to zero are strongly boundary-dominated, as with the couplings (3), for small α . As a bonus, the temporal behavior of the probability to collect the transmitted state at the receiving end of the chain changes from an extremely spiky shape with a needle-like maximum to a single broad and smooth maximum, thus making it much easier to measure the transmitted state at the right instant. This feature can be intuitively understood from the fact that the states dominating the

transfer involve only small energy differences, i.e. long time scales.

The state transfer scheme just discussed can be extended quite naturally, as already suggested in [1]: collecting additional closely spaced eigenvalues in the center of the spectrum generates boundary states on more sites. A contraction like (4) with $(N-5)$ in place of $(N-3)$ and acting on all levels except the previously created triplet generates a quintuplet of levels with unit spacing. A look at the coupling constant distribution shows that the region of nearly constant J_i around $i = \frac{N-1}{2}$ shrinks with every contraction step, giving way to oscillatory behavior near the ends of the chain.

The spatial structure of the energy eigenstates is shown in Fig. 1. The states of the quintuplet clearly show large weights on the two first and last sites of the chain, respectively, while all other states have negligible weight there.

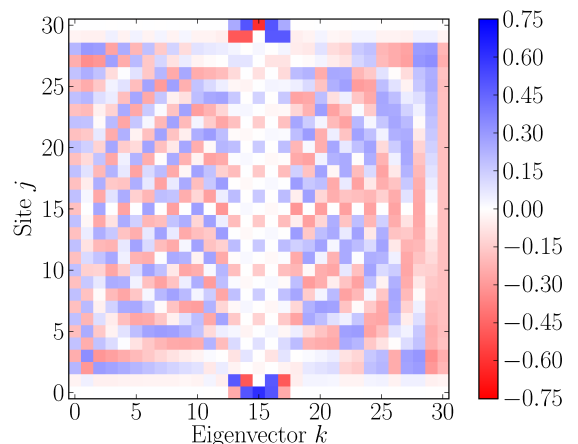


FIG. 1: (Color online) Eigenvectors for a chain with a quintuplet of closely spaced energies, $N = 31$. The states of the quintuplet are basically localized on the two pairs of sites close to the ends of the chain.

The transfer properties of the system were studied for the set of Bell states, which form a maximally entangled basis in the two-spin Hilbert space:

$$|\psi_1\rangle_\pm = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle); |\psi_2\rangle_\pm = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle). \quad (5)$$

In terms of Jordan-Wigner fermions, $|\psi_1\rangle_\pm$ belong to the one-particle subspace of the full chain Hamiltonian (1), while $|\psi_2\rangle_\pm$ contain two-particle components and zero-particle components, the latter having trivial dynamics. We denote by $|\phi_i\rangle$ the state with the first two spins in one of the states (5), with all other spins down, and by $|\phi_f\rangle$ the spatial mirror image of $|\phi_i\rangle$. Then, a convenient measure for the fidelity of transmission can be defined by

$$|f(t)| = |\langle\phi_f|e^{-iHt}|\phi_i\rangle|. \quad (6)$$

Fig. 2 shows $|f(t)|$ for the Bell states $|\psi_1\rangle_{\pm}$ with the $N = 31$ chain discussed above. ($|f(t)|$ is identical for the two states.)

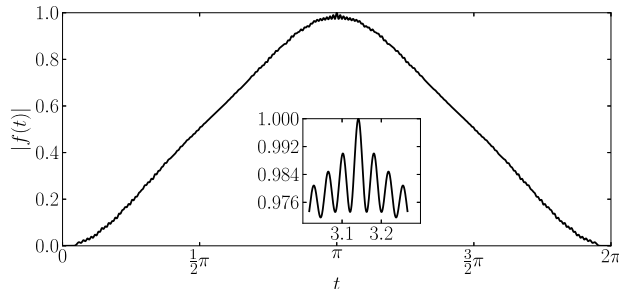


FIG. 2: Fidelity of transmission (6) of the Bell states $|\psi_1\rangle_{\pm}$ for a chain with a quintuplet of closely spaced energies, $N = 31$. The inset shows the region close to the perfect transfer time, $t = \pi$.

Perfect transfer at $t = \pi$ is possible, but small high-frequency oscillations are visible. They are due to the admixture of higher-energy states. The oscillations become stronger for longer chains, making PST extremely difficult beyond $N \approx 60$, because the fidelity maximum at $t = \pi$ becomes extremely narrow. This can be remedied by another contraction of the spectrum. Subtracting $\Delta \text{sgn} \varepsilon_{\nu}$, with $\Delta = 60$, from all energies outside the quintuplet leads to the spectrum $\varepsilon_{\nu} = 0, \pm 1, \pm 2, \pm 7, \pm 70, \dots$ for the $N = 71$ chain, which displays a smooth fidelity vs. time curve, see Fig. 4. The eigenvectors for $N = 71$ and $\Delta = 60$ are shown in Fig. 3. Comparison to Fig. 1

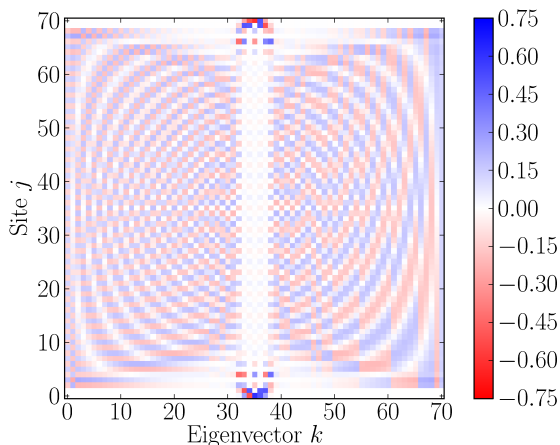


FIG. 3: (Color online) Eigenvectors for a chain with a quintuplet of closely spaced energies, $N = 71$, and an additional contraction by $\Delta = 60$. Note that seven states in the center of the energy spectrum are concentrated near the boundaries, while all other states extend through the whole system.

shows that the quintuplet of boundary-dominated states is on its way to develop into a septuplet. Furthermore it

should be noted that the quintuplet contains almost the complete weight of states localized on the first two lattice sites. This is definitely different for $\Delta = 0$ (not shown here), where the position eigenstate on site 2 contains significant weight from energy eigenstates close to the boundaries of the spectrum. This is what causes small high-frequency oscillations in the transfer fidelity of the single-qubit state initially located at site 2, and consequently, of the Bell states $|\psi_1\rangle_{\pm}$ as shown, for example, in Fig. 2 for $N = 31$.

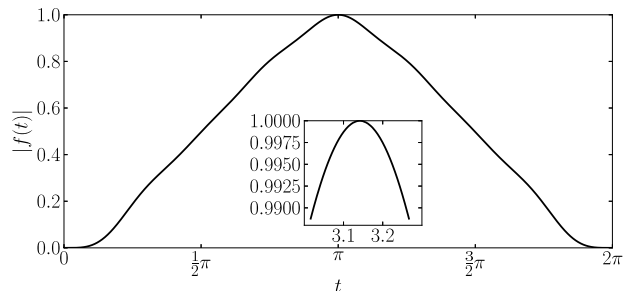


FIG. 4: Fidelity of transmission (6) of the Bell states $|\psi_1\rangle_{\pm}$ for a chain with a quintuplet of closely spaced energies, $N = 71$, and an additional contraction by $\Delta = 60$. The inset shows the region close to the perfect transfer time, $t = \pi$.

The recipe just demonstrated for $N = 71$ is also applicable for longer chains. For $N \approx 200$ the optimal contraction parameter turns out to be $\Delta = N - 7$, meaning that the quintuplet of levels with unit spacing has grown into a septuplet. Continuation of the process generates a nonuplet of states which, for example, leads to decent behavior for $N = 321$, as shown in Fig. 5.

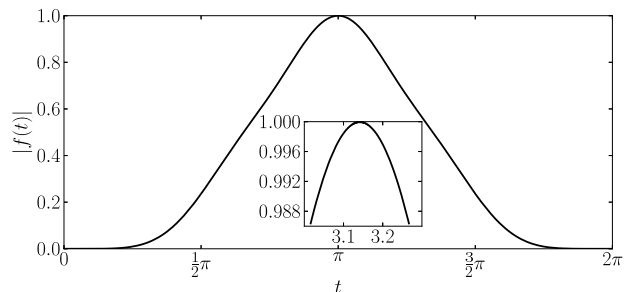


FIG. 5: Fidelity of transmission (6) of the Bell states $|\psi_1\rangle_{\pm}$ for a chain with a nonuplet of closely spaced energies, $N = 321$. The inset shows the region close to the perfect transfer time, $t = \pi$.

The Bell states $|\psi_2\rangle_{\pm}$ involve the dynamics of two-particle states. Since the vacuum component of those states is trivial, the transfer fidelities of $|\psi_2\rangle_{\pm}$ are equal. It turns out that the fidelity of $|\psi_2\rangle_{\pm}$ is slightly more delicate than that of $|\psi_1\rangle_{\pm}$, with narrower maxima and stronger oscillations. Nevertheless, the configuration

with $N = 71$ and additional contraction by $\Delta = 60$ yields similar excellent transfer properties for states $|\psi_1\rangle_\pm$ (Fig. 4) and $|\psi_2\rangle_\pm$ (Fig. 6.)

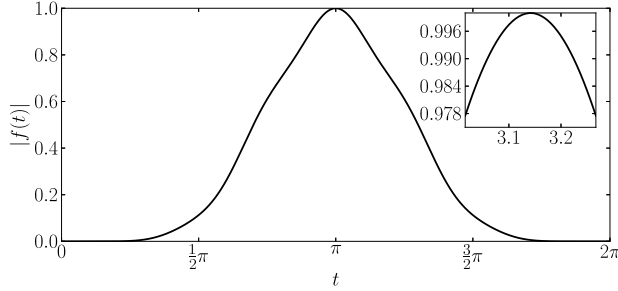


FIG. 6: Fidelity of transmission (6) of the Bell states $|\psi_2\rangle_\pm$ for a chain with a quintuplet of closely spaced energies, $N = 71$, and an additional contraction by $\Delta = 60$. The inset shows the region close to the perfect transfer time, $t = \pi$.

Since all Bell states can be transferred perfectly, one is tempted to conclude that PST is possible for arbitrary two-spin states, the Bell states forming a basis in the two-spin Hilbert space. However, this is not so, due to the presence of different phase factors [19]. To discuss those it is convenient to use a description in terms of Jordan-Wigner fermions. Phase differences occur between states with different numbers of up spins (Jordan-Wigner fermions). Two mechanisms are responsible for these phase differences. Firstly, the perfect transfer of every up spin is accompanied by a fixed phase shift, and secondly, sign changes occur due to the statistics of the Jordan-Wigner fermions.

The phase φ involved in the PST of single-spin (single-fermion) states by mirroring [5–8] is defined by

$$e^{-iH\tau}|i\rangle = e^{i\varphi}|N+1-i\rangle. \quad (7)$$

Here, $|i\rangle$ denotes the state with a single up spin at site i and down spins everywhere else; τ is the perfect transfer time. The phase φ does not depend on the site index i ; for the PST chain with $J_i = \sqrt{i(N-i)}$ [5]

$$\varphi = -\frac{\pi}{2}(N-1). \quad (8)$$

This can be derived from the analogy between the rotation of a spin- S particle in a transverse magnetic field and the motion of a particle along a chain with $2S+1$ sites, each site corresponding to an S_z eigenstate, as discussed, for example, in [20]. It turns out that (8) also holds for general PST chains of odd length N . This follows from the general properties of the eigenvectors and eigenvalues of the tridiagonal single-particle Hamiltonian matrix described earlier. For even N there are two cases which have to be distinguished. Note that for all PST chains, differences between neighboring single-particle energies must be odd numbers (in appropriate units). Thus, for

even N the two smallest (in absolute value) eigenvalues must be $\pm|l + \frac{1}{2}|$ with some integer l , and (8) changes to

$$\varphi = -\frac{\pi}{2}(N-1) - \pi l. \quad (9)$$

To conclude the discussion of the single-particle phase we note that φ may be adjusted by adding a constant magnetic field in z direction, corresponding to a nonzero constant diagonal in the Hamiltonian matrix.

A two-particle state with up spins at sites i and $j > i$ is equivalent to a two-fermion Fock state:

$$|i, j\rangle = c_i^\dagger c_j^\dagger |0\rangle \quad (10)$$

where $|0\rangle$ is the vacuum (all spins down) state. The PST property (7) in combination with Fermi statistics, determines the time evolution of that state:

$$\begin{aligned} e^{-iH\tau}|i, j\rangle &= e^{-iH\tau} c_i^\dagger c_j^\dagger |0\rangle \\ &= e^{2i\varphi} c_{N+1-i}^\dagger c_{N+1-j}^\dagger |0\rangle \\ &= -e^{2i\varphi} c_{N+1-j}^\dagger c_{N+1-i}^\dagger |0\rangle \\ &= e^{i(2\varphi+\pi)} |N+1-j, N+1-i\rangle. \end{aligned} \quad (11)$$

The generalization to a state with n fermions is straightforward; the phase picked up during PST then is

$$\phi_n = n\varphi + n(n-1)\frac{\pi}{2}. \quad (12)$$

For the Bell states this means that $|\psi_1\rangle_\pm$ and $|\psi_2\rangle_\pm$ are transferred with different phase factors, even if $\varphi = 0 \pmod{2\pi}$ can be achieved by choosing $N = 4k+1$, for example. Furthermore, if the spins at sites 3 through N are not initialized to the down state, particle-number dependent phase factors will mix up the transferred state.

However, if initialization is possible, there is a protocol that achieves PST with equal phases for all Bell states. The single-particle phase should be adjusted to $\varphi = \frac{\pi}{2}$ (either by making N even or by an external field). Then ϕ_n is effectively zero for even particle number n and $\frac{\pi}{2}$ for odd n . The first two sites then are initialized to an arbitrary superposition of the four Bell states, while all other sites stay initialized to the down state. Subsequently two controlled- X gates are applied, with qubits 1 and 2 as control and (say) qubit 3 as target qubits: $CX(1,3)CX(2,3)$. Obviously this creates a superposition of states with only even total particle numbers which can be transferred without additional phase factors.

To summarize, we have extended the quantum information transfer scheme given in [1] from single-qubit states to arbitrary two-qubit states by studying the transfer fidelity for all four Bell states for states of up to hundreds of spins, in the absence of built-in disorder. Phase factors causing unwanted interference between Bell states containing even and odd numbers of up spins can be compensated for. Our approach follows the lines already suggested in [1]. The transfer of two-qubit states is achieved

by creating a quintuplet of closely spaced equidistant energy eigenvalues in the center of the spectrum. The corresponding states suffice to expand arbitrary two-qubit states. We have shown how additional manipulations of the energy spectrum may be used to transfer Bell states along progressively longer states. These additional modifications essentially create higher multiplets of energy eigenstates which are concentrated on more boundary sites and may serve to transfer also states of three or more qubits. The protocol compensating for unwanted phase factors may also be naturally extended to states involving three or more spins.

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